Multiple coincidences and the quantum state reconstruction problem

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We analyze the recently proposed method [H. Paul et al., Phys. Rev. Lett. 76, 2464 (1996)] for reconstructing the quantum state of a light field from multiple coincidences measured at the outputs of a passive multiport. We show that applying a large multiport the reconstruction of a pure state becomes possible using avalanche photodiode-type detectors. The presented simulations show that the photon chopping scheme is appropriate for the indirect measurement of the photon statistics of a weak nonclassical signal. [S1050-2947(97)07507-0]

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I. INTRODUCTION

The problem of determining the wave function of a physical system from observable quantities has attracted attention since the dawn of quantum mechanics [1]. Experimental advances in state preparation revived the interest in this issue and successful experiments have been reported about the reconstruction of the quantum state of various physical systems, such as the center-of-mass motion of an ion in a trap [2], a molecular vibrational mode [3], or one mode of a traveling light field [4]. State reconstruction of the electromagnetic field confined in a cavity seems also feasible [5].

In principle, the knowledge of the statistics of two physical quantities (e.g., the quadratures corresponding to the position and the momentum) suffices to find the state of a single-mode light field if it is in a finite superposition of Fock states [6]. In practice, however, even the direct measurement of the photon statistics of nonclassical fields leads to difficulties [7], because of the nonideal behavior of realistic detectors. Moreover, the finite number of measurement data causes restrictions of a more fundamental type [8] valid for any kind of state reconstruction scheme.

There are several recent proposals for the reconstruction of the state of light [4], applying simple homodyning [9], balanced homodyning [10,11], and other techniques [12]. Optical homodyne tomography proved to be also experimentally successful [11]. In this technique, a classical field is mixed with the signal enabling the use of high efficiency linear response photodiodes.

Another way of enhancing the quality of information provided by a measurement is to detect coincidences [13]. In our recently proposed method [14] a symmetric multiport [15,16] chops up the signal and avalanche photodiode-type detectors measure coincidences at the outputs. These kinds of detectors are practically available with quantum efficiency close to unity. On the other hand, their response is not linear, only the presence of photons is indicated by them. The symmetric multiport distributes the incoming photons with equal probability among the outputs producing weak signals for each detector. This already implies that the performance of the method is better for weak signals or equivalently for large multiports. The simulations presented in this paper confirm that indeed the photon statistics of a weak nonclassical signal can be reconstructed to a remarkable accuracy. To get phase information is a more delicate task. For this purpose interference with a reference beam is used. In our previous work [14] more sensitive detectors, distinguishing the one-photon events, were employed to give complete information about the state. We shall discuss here the possibility of using avalanche photodiode-type detectors also for getting phase information about the state.

This paper is structured as follows. In Sec. II we summarize the basic ideas of the method and show how to reconstruct the photon statistics. In Sec. III we describe the details of how to gain phase information. Section IV contains the discussion of realistic effects in the reconstruction of the photon statistics and the results of the numerical simulation for various input states. The results are summarized in the conclusions. The two Appendixes give some technical details used in the main body of the paper.

II. PHOTON STATISTICS

Let us consider an unknown pure state $|\psi\rangle$. This state can be written in the form

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$

where $|n\rangle$ are the Fock states. Our aim is to determine from coincidences the amplitudes and the phases of the expansion coefficients $c_n=|c_n| \exp(i\varphi_n)$. We first show how to determine the amplitudes and then discuss the problem of finding the phases of these coefficients.

The unknown signal is chopped up by using a multiport arrangement, as shown in Fig. 1. The definition of the unitary transformation corresponding to a fully symmetric multiport [15,16] as well as a practical effective realization of it using
Thus any input Fock state \(|n\rangle\) is transformed as

\[
|n\rangle \rightarrow \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{N^n}} \left( \sum_{j=1}^{N} b_j^\dagger \right)^n |0\rangle.
\]

In terms of output photon-number eigenstates Eq. (3) reads

\[
|n\rangle \rightarrow \sqrt{n!} \sum_{k_1+k_2+\ldots+k_N=n} \frac{1}{\sqrt{k_1!k_2!\ldots k_N!}} \times |k_1, k_2, \ldots, k_N\rangle,
\]

where \(|k_1, k_2, \ldots, k_N\rangle\) describes the output with \(k_j\) photons in the channel \(j\).

Let us now assume that we have two types of realistic detectors. Detectors of the first class (let us call them type I detectors) are able to detect only the presence of photons. In practice such detectors are realized by avalanche photodiodes, which have fairly high quantum efficiency. The more sophisticated detectors (type II) can discriminate between zero, one, and more than one photon. In practice, however, type II detectors, i.e., photomultipliers, have rather low quantum efficiency. When we measure coincidences with type II detectors we can limit ourselves to events where each detector indicates the presence of at most one photon. Let the number of the outputs with one photon detected be \(n\). Such an event can be triggered only by the presence of a corresponding \(n\)-photon state at the input. Consequently, the total probability \(w_n^N\) of such coincidences reads

\[
w_n^N = \frac{n!}{N^n} \left| \langle n \rangle \right|^2.
\]

This is a simple relation between the probability of coincidences (that are measured) and the probability of having an \(n\) Fock state as the initial state. One has to keep in mind that the result is valid only up to \(n \leq N\). Because of this limitation it is always desirable to have as large a multiport as possible.

In the case of practically more interesting high efficiency detectors of type I the simple relation (5) is lost. Now we do not know whether the detector click was caused by one or several photons. The probability that \(n\) photons triggered \(m\) simultaneous clicks of the detectors \((n \geq m)\) is expressed as

\[
P_{M,n}^N = \frac{n!}{N^n} \sum_{k_1+k_2+\ldots+k_N=n} \frac{1}{k_1!k_2!\ldots k_N!}. \quad (6)
\]

Note that \(m\) of the indices \(k_i\) have to be nonzero. We can simplify the multiple summation, as is shown in Appendix B, and arrive at the result \((N \geq m)\)

\[
P_{M,n}^N = \frac{1}{N^m} \sum_{m=0}^{m} (-1)^{m-i} \binom{m}{i} (m-i)^n. \quad (7)
\]

This sum is zero for \(n < m\). This also satisfies the recurrence relation

\[
P_{M,n+1}^N = \frac{1}{N} \left[ mP_{M,n}^N + (N-m+1)P_{M-1,n}^N \right]. \quad (8)
\]

which can be easily checked by inserting the solution (7) into (8). The two terms in the recursion express the increase in the number of possibilities when one photon is added: the first refers to the case when the number of coincidences does not change, whereas the second corresponds to the possibility of an additional click.

As it could be expected, the probability for \(m\) clicks gets contributions from all \(n \geq m\). The coincidence probabilities read

\[
w_m^N = \sum_{n=m}^{N} P_{m,n}^N |c_n|^2. \quad (9)
\]

To obtain the photon statistics from the measured coincidences the relation (9) should be inverted. This can be done by inverting the \(N \times N\) matrix \(P_m^N\), which is equivalent to the transformation matrix \(P_{m,n}^N\) truncated so that the elements \(n \geq N\) are set to zero. Thus the inversion takes the form

\[
|c_n|^2 = \sum_{m=n}^{N} I_{m,n,m}^N w_m^N, \quad (10)
\]
The probability to measure a certain state $|\alpha\rangle = \sum_{s=0}^{\infty} \alpha_s |s\rangle$ at another input. The reference beam may be in any well-defined state, e.g., in a weak coherent state. The multiport transform leads to the output (see Appendix A)

$$|\Psi_{\text{out}}\rangle = \sum_{r=0}^{\infty} \frac{c_r \alpha_s}{\sqrt{r! s!} \sqrt{N^r + r}} \left( \sum_{i=0}^{N} b_i^r \right)^t \left( \sum_{i=-N/2+1}^{N/2} b_i^t \right) |0\rangle,$$

or expressed in terms of the multimode Fock states

$$|\Psi_{\text{out}}\rangle = \sum_{r,s} c_r \alpha_s \sqrt{r! s!} \sum_{k_1,\ldots,k_N} \sum_{l_1,\ldots,l_N} \left( \begin{array}{c} + \\ - \end{array} \right) \times \left( \frac{k_1! \cdots k_N! l_1! \cdots l_N!}{k_1! \cdots k_N! l_1! \cdots l_N!} \right) \times |(k_1 + l_1), \ldots, (k_N + l_N)\rangle.$$

We introduced the shortend notation $\binom{s}{r} = (-1)^{s-r} s_r$. The probability to measure a certain state $|q_1, q_2, \ldots, q_N\rangle$ reads
where we used the notation \( n = \sum q_i \) and the conditions \( r + s = n, \ k_i + l_i = q_i \). For the indices \( l_i \) it holds that \( 0 \leq l_i \leq q_i \). The considerations so far are general. When measuring with type II detectors the single-photon events may be selected. The probabilities depend on the choice of detectors, as is indicated by the \( \left( \begin{array}{c} p \nolimits \end{array} \right) \) factor. In our previous paper [14] our aim was just to demonstrate the basic idea of the method, thus we used the selected statistics when only the detectors with \( i \leq N/2 \) ("first") detectors fire, which makes the probabilities independent of the choice of the detectors. Here we give the complete treatment of the problem.

It turns out that the choice of the detectors is essential, not only because of the technicalities of the calculations, but also physically. The crucial point is how many of the detectors with \( i < N/2 \) ("last") detectors give a click. Let us denote with \( p \) the number of them. The statistics of coincidences with at most one photon at one detector is derived by setting a set of the \( q_i \) indices equal to unity, and the others to zero. Collecting all the possible sets we get the coincidence statistics depending on the two relevant parameters: the total number of photons detected \( n \) and the number of firing detectors with \( i > N/2 \), denoted by \( p \),

\[
w^{N}_{n,p} = \left( \begin{array}{c} N/2 \nolimits \end{array} \right) \left( \begin{array}{c} N/2 \nolimits \end{array} \right) N/2 \prod_{i=n}^{p} \frac{1}{N^n} S_{n,p}.
\]

For notational convenience we introduced the "interference factor" \( S_{n,p} \), which does not depend on \( N \)

\[
S_{n,p} = \left( \begin{array}{c} \sum_{s=0}^{n} c_{n-s} \sqrt{(n-s)!s!} C(s,p,n) \nolimits \end{array} \right) ^2.
\]

where

\[
C(s,p,n) = \sum_{i=\text{Min}[s,p]}^{\text{Max}[s,n-p]} (-1)^{p-i} \binom{n-p}{i} \binom{n}{s-i}.
\]

The combinatorial prefactors in Eq. (17) tell in how many ways we can distribute \( p \) and \( n-p \) photons to \( N/2 \) detectors. Equation (16) leads to the factor \( 1/N^n \) and to the interference factor, in which the multiple sum could be simplified using the fact that \( q_i \) and thus \( l_i \) as well can be only 0 or 1. The determination of the phases of the coefficients \( c_i \) can be carried out iteratively. The amplitudes are assumed to be determined by applying the results of the previous section. The \( w_{n,p} \) probabilities contain information about the first \( n \) unknown phases, thus starting from \( n = 1 \) and setting the phase of \( c_0 \) to an arbitrary value the phase of \( c_1 \) can be expressed. To illustrate the importance of distinguishing between the "first" and "last" detectors let us examine explicitly what happens if we add the two terms \( w^N_{1,0} \) and \( w^N_{1,1} \):

\[
w^N_{1,0} + w^N_{1,1} = \frac{1}{2} \left| c_1 \alpha_0 + c_0 \alpha_1 \right|^2 + \frac{1}{2} \left| c_1 \alpha_0 - c_0 \alpha_1 \right|^2 = \left| c_1 \alpha_0 \right|^2 + \left| c_0 \alpha_1 \right|^2.
\]

The terms \( 1/2N \) result from the fact that we have to take into account the rare (suppressed by \( 1/N \)) possibility that two photons go to one detector. We see again that, except for a small contribution proportional to \( 1/N \), all the terms that could give phase information cancel. Again, we have to consider separately all the cases with different \( p \).

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We have presented a method to obtain phase information of a pure state by detecting single-photon coincidences. In the following we discuss how the scheme changes when realistic, type I detectors are applied.

### B. Reconstructing a pure state with type I detectors

The type I detectors can distinguish only between the presence and absence of photons. Thus \( m \) firing detectors means that there can be any number \( n \geq m \) of photons present. In considering the photon statistics only, we can invert the data obtained with type I detectors to get the original photon statistics. A similar procedure for the determination of the phases would be, however, very difficult. Fortunately, by assuming that we apply a large multiport \( (N \) is large), we can approach the problem by expanding the coincidence probabilities as a series in \( 1/N \).

For a large multiport, i.e., when the number of detectors is comparable to the mean photon number of the signal, the probability to have more than one photon at a detector is small; if we could say that it is negligible, the detectors of type I and II would not differ essentially. If we do not neglect the clicks caused by many photons we find for the probability of \( m \) coincidences the sum

\[
w^N_{m,p} = w^N_{m,m,p} + w^N_{m,m+1,p} + w^N_{m,m+2,p} + \cdots,
\]

where the second index in \( w^N_{m,n,p} \) refers to the number of photons present. The subsequent terms differ by a factor of \( O(1/N) \) as it follows from Eq. (16). The first term is equivalent to the result of the previous chapter. We will proceed by calculating the second term, giving a first-order correction.
This corresponds to the case when one of the detectors is hit by two photons. It means one has to evaluate Eq. (16) with the condition that one of the indices \( q_i \) equals 2 (let us denote its index with \( x \)), while the others are 1 or 0. The simplification of the multiple sum over the combinatorial factors in Eq. (16) is not that simple as it was in the previous section, it also depends on whether the extra photon hits the “first” or “last” detectors. In the case it hits one of the “first” detectors the simplification leads to

\[
\sum_{x=1}^{\infty} \frac{\sqrt{2}}{(2-l_x)!l_x!} C(s,p,m+1), \quad x \leq \frac{N}{2}.
\]

(23)

Thus we get the half of the interference factor in the previous section [see Eq. (18)]. In the second case \((x > N/2)\) the simplification leads to the interference factor of the following form:

\[
S'_{m,p} = \sum_{s=0}^{m} c_{m-s} \sqrt{(m-s)!s!} C(s,p,m) - 4 \sum_{s=1}^{m} c_{m-s} \sqrt{(m-s)!s!} C(s-1,p,m-2).
\]

(24)

Taking into account all the possible values for \( x \) we arrive at the coincidence statistics with first-order correction

\[
w_{m,p} = \left( \frac{N/2}{p} \right) \left( \frac{m-p}{N-m} \right) S_{m,p} + \frac{1}{N} \left( \frac{1}{2} (m-p) S_{m+1,p} + \frac{1}{2} p S'_{m+1,p-1} \right) + O(1/N^2).
\]

(25)

The higher-order terms would contain combinatorial factors representing the different possibilities of adding 2, 3, 4, etc. photons to the “first” and “last” detectors, multiplied by corresponding interference factors. The probabilities \( w_{1,p} \) contain, because of the first-order correction term, also the coefficient \( c_2 \) whose phase is unknown. Thus arg(\( c_2 \)) cannot be solved uniquely with only two reference states \( |\alpha\rangle \) as in the case of type II detectors, but we have to use additional reference beams with different phase properties.

### C. Reconstruction of a density matrix

When the unknown signal is in a mixed state, we describe it by the density matrix

\[
\hat{\rho} = \sum_{r,r'=0}^{\infty} \frac{c_{rr'}}{\sqrt{r!r'!}} (a_r^\dagger)^r (0) (a_{r'})^{r'}.
\]

(26)

The reference beam fed into the \( N/2 + 1 \)st input can also be in a mixed state

\[
\hat{\rho}_{\text{ref}} = \sum_{s,s'=0}^{\infty} \frac{\alpha_{ss'}}{\sqrt{s!s'!}} (a_{N/2+1})^s (0) (a_{N/2+1})^{s'}.
\]

(27)

The density matrix describing the output of the multiport is now

\[
\hat{\rho}_{\text{out}} = \sum_{r,r'=0}^{\infty} \sum_{s,s'=0}^{\infty} \frac{c_{rr'} \alpha_{ss'}}{\sqrt{r!r'!s!s'!}} \frac{1}{\sqrt{N+r+r'+s+s'}} \left( \sum_{i=1}^{N/2} b_i^r \right) \left( \sum_{i=1}^{N} b_i^s \right) \left( \sum_{i=N/2+1}^{N} b_i^r \right) \left( \sum_{i=N+1}^{N/2+1} b_i^s \right) (0) (0) \left( \sum_{i=1}^{N/2} b_i^{r'} \right) \left( \sum_{i=1}^{N} b_i^{s'} \right) \left( \sum_{i=N/2+1}^{N} b_i^{r'} \right) \left( \sum_{i=N+1}^{N/2+1} b_i^{s'} \right).
\]

(28)

The probability to get \( \{q_1, q_2, \ldots, q_N\} \) photons at the corresponding detectors is \( \langle q_1, q_2, \ldots, q_N | \rho_{\text{out}} | q_1, q_2, \ldots, q_N \rangle \). From the both sides of the projector \( (0) (0) \) in \( \hat{\rho}_{\text{out}} \) we get two summations containing the factor \( (\cdot)^{n-p} \). To each of these we can apply all the operations that were performed in the pure state case. The result both in the case of type II as well as in type I detectors is exactly the same as above, except that the “interference factors” \( S'_{n,p} \) are now defined as

\[
S'_{n,p} = \sum_{s,s'=0}^{\infty} \frac{c_{n-s,n-s'} \alpha_{ss'}}{\sqrt{(n-s)!s!(n-s')!s'!}} \left[ \sum_{i=\text{Max}(0,p-(n-s))}^{\text{Min}(i,p)} (-1)^{p-i} \binom{n-p}{s-i} \right] \left[ \sum_{i=\text{Max}(0,p-(n-s'))}^{\text{Min}(i',p)} (-1)^{p-i'} \binom{n-p}{s'-i} \right]
\]

(29)
for $S_{n,p}$ correspondingly. For example, for the one-photon coincidence we have

$$S_{1,0} = c_{11} \alpha_{00} + c_{10} \alpha_{01} + c_{01} \alpha_{10} + c_{10} \alpha_{11},$$

$$S_{1,1} = c_{11} \alpha_{00} - c_{10} \alpha_{01} - c_{01} \alpha_{10} + c_{10} \alpha_{11}. \quad (30)$$

Another way to obtain the “interference factors” is just to replace $c_i c_j^*$ in the pure state case by $c_{ij}$ (and $\alpha_i \alpha_j^*$ by $\alpha_{ij}$, if the reference beam is also in a mixed state).

Mathematically the treatment in the mixed state case differs very little from that for the pure state. The essential difference comes from the fact that now $|c_{ij}|$ can deviate from $|c_i||c_j|$ when $i \neq j$. Thus we have more unknown variables contributing to the probabilities, and are forced to use extra reference beams (reference phases). The maximum number of reference beams is limited by the cutoff in the photon number. (Related to this see [19], where it was investigated how many tomographic phase angles are needed to reconstruct a density matrix truncated at $n_{\text{max}}$.)

**IV. RECONSTRUCTION OF THE PHOTON-NUMBER DISTRIBUTION: NUMERICAL SIMULATION**

In the previous section we focused our attention on the principal aspects of how one can reconstruct an unknown state from the measured coincidences. In this section we consider a realistic situation and present numerical simulations of the photon-number distribution (PND) reconstruction.

In a realistic situation three main sources of errors should be taken into account: finite-size effects due to the use of a finite multiphoton, statistical errors, and detector inefficiencies (losses). In order to see clearly their effect we treat them separately. The size of the multiphoton ($N$, number of outputs) should be compared to the width of the initial photon-number distribution. In the limiting case of an infinite number of outputs all the photons arrive at different detectors and the photon statistics can be measured directly just by counting the photons in coincidence. For a finite distribution with $n_{\text{max}} \leq N$ the correspondence between the coincidences and the photon statistics is one to one. If, however, $n_{\text{max}}$ exceeds $N$ or the distribution is infinite, the inversion of the truncated matrix will not yield the exact result anymore. To estimate the error caused by the tail of the distribution one has to analyze the effect of the partial inversion of the multiphoton transform in Eq. (11). Applying the transformation and the truncated inversion in sequence the reconstructed distribution reads

$$|c_i|^{\text{rec}}|^2 = \sum_{n=0}^{\infty} \gamma_{i,n} |c_n|^2 = |c_i|^2 + \sum_{n=N+1}^{\infty} \gamma_{i,n} |c_n|^2, \quad (32)$$

where

$$\gamma_{i,n} = \sum_{m=0}^{N} t_{i,m}^N P_{m,n}. \quad (33)$$

In the $n \to \infty$ limit we find

$$\lim_{n \to \infty} P_{m,n} = \lim_{n \to \infty} \left( \frac{N}{m} \sum_{i=0}^{m} (-1)^i \left( \frac{m}{N} \right)^i \right) = \delta_{m,N}, \quad (34)$$

and thus

$$\lim_{n \to \infty} \gamma_{i,n} = t_{i,N}^N, \quad (35)$$

where the convergence is monotonic. Using Eq. (35), an upper limit for the absolute value of the error can be obtained in the form

$$||c_i|^{\text{rec}}|^2 - |c_i|^2| \leq ||t_{i,N}^N| \sum_{n=N+1}^{\infty} |c_n|^2. \quad (36)$$

The limit depends then on the sum of the terms of the original photon distribution in the “tail,” verifying that a large enough multiphoton minimizes this type of error.

In the simulations we have assumed the detectors to be of type I (high quantum efficiency). We have chosen three representative examples of possible initial PND’s (a single Fock state, a finite superposition of Fock states, and a weakly squeezed vacuum) and performed numerical simulations of the proposed experiments. The simulation is realized in three steps. First we numerically generate events (coincidences), then we calculate from the generated coincidences the (uncorrected) PND. Finally we perform the Bernoulli transformation on the uncorrected PND, which compensates the losses of the detectors [20]. We investigated how the reconstructed PND’s are influenced by the size of the multiphoton $N$, amount of data (number of single coincidences), and the detector efficiency $\eta$. As a measure of the similarity between the reconstructed and the original PND we evaluate the distance

$$d(\hat{\rho}, \hat{\rho}') = \text{Tr}((\hat{\rho} - \hat{\rho}')^2), \quad (37)$$

where $\hat{\rho}$ is the PND matrix of the original and $\hat{\rho}'$ of the reconstructed state. The parameter $d(\hat{\rho}, \hat{\rho}')$ has a maximum value of 2, and it tends to zero when the two matrices become equal.

As a first example we reconstruct the photon-number distribution of a five-photon Fock state [5]. We choose to use the detector efficiency $\eta = 0.9$, and vary the number of output channels from $N = 10$ to $N = 30$ and the number of coincidences from $10^4$ to $10^6$. The results are summarized in Table I. The reconstructed PND coefficients without the loss correction are denoted as $P(j)$ and as $P(j)'$ after performing the loss correction. The reconstructed PND without detector loss correction, $P(j)$, improves slightly when the size of the multiphoton is increased from $N = 10$ to $N = 30$. The reconstructed PND suggests at this stage non-negligible contributions from the states [4] and [5]. The results improve significantly after the loss correction is performed. Now with increasing $N$ and the number of runs the result becomes nearly perfect. Note that some of the probabilities $P(j)'$ are negative. This originates from the numerical instability of the procedure, and can be eliminated by a more careful numerical treatment of the problem.
TABLE I. The results of the numerical simulation for the initial Fock state $|5\rangle$. $P(j)$ is the reconstructed PND without and $P(j)'$ with the detector loss correction. The detector efficiency was set to $\eta=0.9$.

<table>
<thead>
<tr>
<th>Number of ports</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>30</th>
<th>30</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of runs</td>
<td>$10^4$</td>
<td>$10^5$</td>
<td>$10^6$</td>
<td>$10^4$</td>
<td>$10^5$</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Distance</td>
<td>0.293</td>
<td>0.293</td>
<td>0.284</td>
<td>0.265</td>
<td>0.279</td>
<td>0.279</td>
</tr>
<tr>
<td>Corr. distance</td>
<td>$5.7 \times 10^{-4}$</td>
<td>$8.7 \times 10^{-4}$</td>
<td>$9 \times 10^{-5}$</td>
<td>$1.4 \times 10^{-3}$</td>
<td>$1.6 \times 10^{-5}$</td>
<td>$6.9 \times 10^{-6}$</td>
</tr>
<tr>
<td>$P(0)$</td>
<td>0.0</td>
<td>0.0</td>
<td>$1 \times 10^{-5}$</td>
<td>0.0</td>
<td>0.0</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>$P(1)$</td>
<td>$2.3 \times 10^{-4}$</td>
<td>$3.6 \times 10^{-4}$</td>
<td>$4.2 \times 10^{-4}$</td>
<td>$2.5 \times 10^{-4}$</td>
<td>$4.1 \times 10^{-4}$</td>
<td>$4.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>$P(2)$</td>
<td>$8.0 \times 10^{-3}$</td>
<td>$9.6 \times 10^{-3}$</td>
<td>$8.4 \times 10^{-3}$</td>
<td>$7.5 \times 10^{-3}$</td>
<td>$8.7 \times 10^{-3}$</td>
<td>$8.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$P(3)$</td>
<td>$7.0 \times 10^{-2}$</td>
<td>$6.7 \times 10^{-2}$</td>
<td>$7 \times 10^{-2}$</td>
<td>$7.83 \times 10^{-2}$</td>
<td>$7.2 \times 10^{-2}$</td>
<td>$7.2 \times 10^{-2}$</td>
</tr>
<tr>
<td>$P(4)$</td>
<td>0.336</td>
<td>0.338</td>
<td>0.331</td>
<td>0.313</td>
<td>0.326</td>
<td>0.328</td>
</tr>
<tr>
<td>$P(5)$</td>
<td>0.582</td>
<td>0.584</td>
<td>0.589</td>
<td>0.599</td>
<td>0.592</td>
<td>0.592</td>
</tr>
<tr>
<td>$P(6)$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P(0)'$</td>
<td>$6.6 \times 10^{-5}$</td>
<td>$2.7 \times 10^{-5}$</td>
<td>$1.1 \times 10^{-5}$</td>
<td>$-5.23 \times 10^{-6}$</td>
<td>$-2.48 \times 10^{-6}$</td>
<td>$-9.75 \times 10^{-7}$</td>
</tr>
<tr>
<td>$P(1)'$</td>
<td>$-7.0 \times 10^{-4}$</td>
<td>$-7.7 \times 10^{-4}$</td>
<td>$-2.5 \times 10^{-4}$</td>
<td>$2.5 \times 10^{-4}$</td>
<td>$-2.1 \times 10^{-4}$</td>
<td>$-5.0 \times 10^{-6}$</td>
</tr>
<tr>
<td>$P(2)'$</td>
<td>$6.5 \times 10^{-4}$</td>
<td>$5.26 \times 10^{-3}$</td>
<td>$1.9 \times 10^{-3}$</td>
<td>$-4.5 \times 10^{-3}$</td>
<td>$9.0 \times 10^{-4}$</td>
<td>$2.6 \times 10^{-4}$</td>
</tr>
<tr>
<td>$P(3)'$</td>
<td>$-5.7 \times 10^{-3}$</td>
<td>$-1.52 \times 10^{-2}$</td>
<td>$-6.1 \times 10^{-3}$</td>
<td>$1.77 \times 10^{-2}$</td>
<td>$1.35 \times 10^{-4}$</td>
<td>$-1.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>$P(4)'$</td>
<td>$1.9 \times 10^{-2}$</td>
<td>$2.2 \times 10^{-2}$</td>
<td>$6.9 \times 10^{-3}$</td>
<td>$-2.95 \times 10^{-2}$</td>
<td>$-3.2 \times 10^{-3}$</td>
<td>$1 \times 10^{-3}$</td>
</tr>
<tr>
<td>$P(5)'$</td>
<td>0.98</td>
<td>0.988</td>
<td>0.997</td>
<td>1.01</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$P(6)'$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

As the second example we consider a finite superposition of Fock states; i.e., beginning from a certain $n>N$ all the states $|n\rangle$ are unpopulated. For the simulation we have chosen a state with the following initial photon number distribution:

\[
P(0) P(1) P(2) P(3) P(4) P(5) P(6) \quad 0.1 \quad 0.1 \quad 0.2 \quad 0.1 \quad 0.1 \quad 0.2.
\]

Table II summarizes the results. In the simulation we used two detector efficiencies, $\eta=0.9$ (left part of the table) and $\eta=0.7$ (right part of the table). Already without the loss correction the first terms of the distribution are reproduced to a reasonable extent. After performing the loss correction we get an extremely good approximation for the initial distribution. When the detector efficiency is decreased the results before the correction become worse, but the correction still works very well. Note that the results are already good for a fairly small number of outputs $N=10$.

The last example we have chosen for the simulation is a single mode squeezed vacuum state

\[
|\xi\rangle = \mathcal{N} \exp(\xi \hat{a}^{\dagger 2}) |0\rangle,
\]

where $\mathcal{N}$ is the normalization factor. The mean photon number was set to $\langle n \rangle = 0.2$. The first few nonzero coefficients in the original photon number distribution are

\[
P(0) P(2) P(4) P(6) P(8) \quad 0.913 \quad 7.6 \times 10^{-2} \quad 9.5 \times 10^{-3} \quad 1.3 \times 10^{-3} \quad 1.9 \times 10^{-3}.
\]

The detector efficiency was set to $\eta=0.9$. The numerical results are presented in Table III. This case is especially interesting because the initial PND is in principle an infinite superposition of Fock states. Due to the finite number of output ports the set of coincidence data is truncated and the reconstruction starts from this incomplete data set. This is reflected in the data in Table III. With the increase of $N$ and consequently the cutoff the resolution of the scheme becomes better. However, with higher-dimensional matrices we are also faced with purely numerical problems (inversion of the matrices) and this leads to the negativities of some of the calculated probabilities.

Our numerical testing leads us to the following conclusions. For excitation numbers small compared to the number of outputs, or for finite superpositions of Fock states the method is able to reproduce the unknown superposition coefficients of the state with a sufficient precision. The accuracy of the method can be enhanced by increasing the number of output ports and by performing enough measurements, i.e., by detecting enough coincidences. The number of measurements plays a crucial role mainly when we perform the detection loss correction step in the calculations. When the PND to be detected has a large width compared to the multiport size the first few coefficients can be reproduced.

V. CONCLUSIONS

We analyzed the possibility to reconstruct the quantum state of a single-mode light field from coincidences at the outputs of a symmetric multiport. The correspondence between the signal photon statistics and the output coincidence statistics, measured with realistic detectors (avalanche photodiodes), is one to one if the number of outputs $N$ is larger than the maximum photon number $n_{\text{max}}$. We simulated a realistic measurement process taking into account nonideal detection efficiency, and were able to recover the nonclassical oscillations in the photon statistics of a weak squeezed vacuum signal (mean photon number is less than one). The effect of the tail of a broad photon distribution ($n_{\text{max}}>N$) can be estimated based on the total probability contained in the tail, independently of its shape.
MULTIPLE COINCIDENCES AND THE QUANTUM STATE . . .

The full quantum state can be reconstructed from the interference between the signal and a reference beam by considering single-photon coincidences at the outputs. In the case of avalanche photodiodes, which give information only about the presence of photons, one can expand the probability of coincidences as a series in I/N. The quantum state can be reconstructed from the lowest-order terms. Thus the requirement of having a large multiport, or equivalently a weak signal, is even more fundamental in this case.

TABLE II. The results of the simulation for the initial state (38). The first four columns correspond to the detector efficiency η = 0.9, the second four to η = 0.7.

<table>
<thead>
<tr>
<th>Number of ports</th>
<th>10</th>
<th>20</th>
<th>20</th>
<th>30</th>
<th>10</th>
<th>20</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of runs</td>
<td>10⁵</td>
<td>10⁵</td>
<td>10⁵</td>
<td>10⁶</td>
<td>10⁶</td>
<td>10⁶</td>
<td>10⁶</td>
<td>10⁶</td>
</tr>
<tr>
<td>Distance</td>
<td>1.2×10⁻²</td>
<td>1.1×10⁻²</td>
<td>1.1×10⁻²</td>
<td>1.1×10⁻²</td>
<td>4.8×10⁻²</td>
<td>4.8×10⁻²</td>
<td>4.8×10⁻²</td>
<td>4.8×10⁻²</td>
</tr>
<tr>
<td>Corr. distance</td>
<td>8.2×10⁻⁵</td>
<td>3.9×10⁻⁵</td>
<td>6.5×10⁻⁶</td>
<td>9.5×10⁻⁷</td>
<td>1.0×10⁻⁵</td>
<td>5.0×10⁻⁴</td>
<td>1.3×10⁻⁵</td>
<td>1.1×10⁻⁵</td>
</tr>
<tr>
<td>P(0)</td>
<td>0.11</td>
<td>0.11</td>
<td>0.11</td>
<td>0.112</td>
<td>0.154</td>
<td>0.154</td>
<td>0.154</td>
<td>0.154</td>
</tr>
<tr>
<td>P(1)</td>
<td>0.13</td>
<td>0.13</td>
<td>0.13</td>
<td>0.131</td>
<td>0.205</td>
<td>0.205</td>
<td>0.205</td>
<td>0.203</td>
</tr>
<tr>
<td>P(2)</td>
<td>0.21</td>
<td>0.22</td>
<td>0.217</td>
<td>0.216</td>
<td>0.236</td>
<td>0.236</td>
<td>0.237</td>
<td>0.238</td>
</tr>
<tr>
<td>P(3)</td>
<td>0.18</td>
<td>0.18</td>
<td>0.186</td>
<td>0.185</td>
<td>0.179</td>
<td>0.179</td>
<td>0.178</td>
<td>0.178</td>
</tr>
<tr>
<td>P(4)</td>
<td>0.12</td>
<td>0.116</td>
<td>0.118</td>
<td>0.118</td>
<td>0.126</td>
<td>0.124</td>
<td>0.124</td>
<td>0.125</td>
</tr>
<tr>
<td>P(5)</td>
<td>0.128</td>
<td>0.131</td>
<td>0.129</td>
<td>0.13</td>
<td>0.0764</td>
<td>0.076</td>
<td>0.077</td>
<td>0.0774</td>
</tr>
<tr>
<td>P(6)</td>
<td>0.107</td>
<td>0.106</td>
<td>0.107</td>
<td>0.107</td>
<td>0.0236</td>
<td>0.0239</td>
<td>0.0236</td>
<td>0.0236</td>
</tr>
<tr>
<td>P(7)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

| P(0) ′          | 0.098| 0.0988| 0.0997| 0.0997| 0.098| 0.0986| 0.1| 0.1|
| P(1) ′          | 0.1| 0.1| 0.0994| 0.0996| 0.105| 0.103| 0.0995| 0.09987|
| P(2) ′          | 0.2| 0.201| 0.201| 0.2| 0.1965| 0.197| 0.198| 0.198|
| P(3) ′          | 0.195| 0.201| 0.2| 0.1998| 0.1968| 0.195| 0.202| 0.198|
| P(4) ′          | 0.105| 0.954| 0.1| 0.0999| 0.106| 0.115| 0.098| 0.101|
| P(5) ′          | 0.097| 0.104| 0.0983| 0.1| 0.102| 0.087| 0.1| 0.099|
| P(6) ′          | 0.2| 0.199| 0.201| 0.20| 0.196| 0.203| 0.2| 0.2|
| P(7) ′          | 0| 0| 0| 0| 0| 0| 0| 0|

ACKNOWLEDGMENTS

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APPENDIX A

For reconstructing the photon statistics we need a multiport that realizes the transformation

TABLE III. The results of the simulation for the squeezed vacuum state (39) with the mean photon number ⟨n⟩ = 0.2. The detector efficiency was set to η = 0.9.

<table>
<thead>
<tr>
<th>Number of ports</th>
<th>10</th>
<th>10</th>
<th>10</th>
<th>30</th>
<th>30</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of runs</td>
<td>10⁴</td>
<td>10⁴</td>
<td>10⁴</td>
<td>10⁶</td>
<td>10⁶</td>
<td>10⁶</td>
</tr>
<tr>
<td>Distance</td>
<td>4.6×10⁻⁴</td>
<td>4.2×10⁻⁴</td>
<td>4.0×10⁻⁴</td>
<td>4.8×10⁻⁴</td>
<td>4.0×10⁻⁴</td>
<td>4.1×10⁻⁴</td>
</tr>
<tr>
<td>Corr. distance</td>
<td>2.8×10⁻⁴</td>
<td>2.7×10⁻⁴</td>
<td>7.1×10⁻⁵</td>
<td>1.4×10⁻⁵</td>
<td>7.7×10⁻⁷</td>
<td>2.4×10⁻⁷</td>
</tr>
<tr>
<td>P(0)</td>
<td>0.912</td>
<td>0.913</td>
<td>0.914</td>
<td>0.912</td>
<td>0.913</td>
<td>0.914</td>
</tr>
<tr>
<td>P(1)</td>
<td>1.4×10⁻²</td>
<td>1.4×10⁻²</td>
<td>1.3×10⁻²</td>
<td>1.5×10⁻²</td>
<td>1.4×10⁻²</td>
<td>1.4×10⁻²</td>
</tr>
<tr>
<td>P(2)</td>
<td>6.2×10⁻²</td>
<td>6.2×10⁻²</td>
<td>6.2×10⁻²</td>
<td>6.1×10⁻²</td>
<td>6.2×10⁻²</td>
<td>6.2×10⁻²</td>
</tr>
<tr>
<td>P(3)</td>
<td>2.9×10⁻³</td>
<td>2.6×10⁻³</td>
<td>2.4×10⁻³</td>
<td>2.9×10⁻³</td>
<td>2.7×10⁻³</td>
<td>2.7×10⁻³</td>
</tr>
<tr>
<td>P(4)</td>
<td>6.2×10⁻³</td>
<td>6.1×10⁻³</td>
<td>6.6×10⁻³</td>
<td>6.8×10⁻³</td>
<td>6.4×10⁻³</td>
<td>6.3×10⁻³</td>
</tr>
<tr>
<td>P(5)</td>
<td>4.2×10⁻³</td>
<td>1.7×10⁻³</td>
<td>3.4×10⁻³</td>
<td>1.5×10⁻³</td>
<td>6.2×10⁻⁴</td>
<td>4.1×10⁻⁴</td>
</tr>
<tr>
<td>P(6)</td>
<td>−3.4×10⁻³</td>
<td>−4.5×10⁻⁴</td>
<td>1.1×10⁻³</td>
<td>2.7×10⁻⁴</td>
<td>6.0×10⁻⁴</td>
<td>7.8×10⁻⁴</td>
</tr>
<tr>
<td>P(7)</td>
<td>1.7×10⁻³</td>
<td>4.96×10⁻⁴</td>
<td>−4.1×10⁻⁴</td>
<td>−2.6×10⁻⁴</td>
<td>−1.7×10⁻⁷</td>
<td>5.1×10⁻⁵</td>
</tr>
<tr>
<td>P(0) ′</td>
<td>0.911</td>
<td>0.912</td>
<td>0.913</td>
<td>0.911</td>
<td>0.912</td>
<td>0.913</td>
</tr>
<tr>
<td>P(1) ′</td>
<td>6.2×10⁻³</td>
<td>6.7×10⁻³</td>
<td>6.7×10⁻³</td>
<td>1.98×10⁻³</td>
<td>8.1×10⁻⁴</td>
<td>6.8×10⁻⁴</td>
</tr>
<tr>
<td>P(2) ′</td>
<td>7.6×10⁻²</td>
<td>7.6×10⁻²</td>
<td>7.6×10⁻²</td>
<td>7.4×10⁻²</td>
<td>7.7×10⁻²</td>
<td>7.6×10⁻²</td>
</tr>
<tr>
<td>P(3) ′</td>
<td>9.9×10⁻⁴</td>
<td>1.1×10⁻⁴</td>
<td>−7.4×10⁻⁴</td>
<td>1.28×10⁻⁴</td>
<td>−8.9×10⁻⁵</td>
<td>−6.4×10⁻⁵</td>
</tr>
<tr>
<td>P(4) ′</td>
<td>4.7×10⁻³</td>
<td>7.7×10⁻³</td>
<td>1.0×10⁻²</td>
<td>9.17×10⁻³</td>
<td>9.4×10⁻³</td>
<td>9.5×10⁻³</td>
</tr>
<tr>
<td>P(5) ′</td>
<td>1.1×10⁻²</td>
<td>3.59×10⁻³</td>
<td>−9.4×10⁻⁴</td>
<td>2.1×10⁻³</td>
<td>3.6×10⁻⁴</td>
<td>−1.9×10⁻⁴</td>
</tr>
<tr>
<td>P(6) ′</td>
<td>−8.95×10⁻³</td>
<td>−1.57×10⁻³</td>
<td>2.97×10⁻³</td>
<td>1.07×10⁻³</td>
<td>1.18×10⁻³</td>
<td>1.47×10⁻³</td>
</tr>
<tr>
<td>P(7) ′</td>
<td>3.45×10⁻³</td>
<td>1.04×10⁻³</td>
<td>−1.48×10⁻³</td>
<td>−1.06×10⁻⁴</td>
<td>−1.39×10⁻⁴</td>
<td>−1.2×10⁻⁵</td>
</tr>
</tbody>
</table>
between the first input creation operator and the output creation operators. For this we do not need a fully symmetric multiport [15,16], but a more efficient realization using a plate beam splitter is sufficient [17,18].

The plate beam splitter is a simple device that splits the incoming (coherent) beam into $N$ equal parts. The corresponding transformation is

\[(U_N)_{i,j} = \sqrt{\frac{1}{N}}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq N\]

(A2)

\[(U_N)_{i,j} = -\sqrt{\frac{1}{(N+2-j)(N+1-j)}}, \quad 1 \leq j < i.

(A3)

\[(U_N)_{i,i+1} = \sqrt{\frac{N-i}{N+1-i}},

(A4)

\[(U_N)_{i,j} = 0, \quad j > i + 1.

(A5)

This orthogonal transformation is equivalent to an array of $N$ beam splitters of properly chosen transmittivity. Such devices are available in the form of a glass plate with modulated transmittivity along the direction of the incoming beam propagation [17]. Only one plate beam splitter of dimension $N$ is needed in reconstructing the photon statistics. The field in the unknown state has to be fed into the first input of the plate, the corresponding creation operator transforms according to Eq. (A1).

In order to reconstruct the phases, we need interference between the unknown beam and a reference beam; we require that the first input creation operator transforms like Eq. (A1), and the $N/2 + 1$st like

\[a_{N/2+1} \rightarrow \frac{1}{\sqrt{N}} \sum_{j=1}^{N/2} b_j - \sum_{j=N/2+1}^{N} b_j.\]

(A6)

This can be realized by combining the outputs of two plates of dimension $N/2$ by balanced beam splitters. The unknown beam is fed into the first input of one of the plates and the reference beam into the first input of the other plate. The transformation describing this device is simply a combination of the two-plate beam-splitter transformations and the one describing the balanced beam splitters:

\[
\bar{U}_N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1_{N/2} & 1_{N/2} \\ 1_{N/2} & -1_{N/2} \end{bmatrix} \begin{bmatrix} U_{N/2} & 0 \\ 0 & U_{N/2} \end{bmatrix}
\]

\[
= \begin{bmatrix}
\sqrt{\frac{1}{N}} & \sqrt{\frac{N/2-1}{N}} & 0 & \ldots & \sqrt{\frac{1}{N}} & \sqrt{\frac{N/2-1}{N}} & 0 & \ldots \\
\sqrt{-\frac{1}{N(N/2-1)}} & \sqrt{\frac{N/2-2}{N-2}} & \ldots & \sqrt{-\frac{1}{N(N/2-1)}} & \sqrt{\frac{N/2-2}{N-2}} & \ldots & \ldots & \ldots \\
\sqrt{\frac{1}{N}} & \sqrt{\frac{N/2-1}{N}} & 0 & \ldots & -\sqrt{\frac{1}{N}} & -\sqrt{\frac{N/2-1}{N}} & 0 & \ldots \\
\sqrt{-\frac{1}{N(N/2-1)}} & \sqrt{\frac{N/2-2}{N-2}} & \ldots & -\sqrt{\frac{1}{N(N/2-1)}} & \sqrt{\frac{1}{N}} & \sqrt{\frac{N/2-2}{N-2}} & \ldots & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix}.
\]

(A7)

Note that only the first and $N/2 + 1$st columns of this transformation are relevant to the reconstruction scheme. The number of beam splitters needed becomes $3N/2$. The numbers of individual components required for the reconstruction of the photon statistics ($N$) and the phases ($3N/2$) are much smaller than the general upper estimate for an arbitrary unitary transform $[N(N-1)/2]$ [15].

**APPENDIX B**

We consider the problem of calculating the following sum of multinomial coefficients:

\[S^m_{N,m} = \sum_{k_1 + k_2 + \ldots + k_m = n} \frac{n!}{k_1! \ldots k_m!}, \quad N \geq n \geq m, \quad (B1)\]

where $(m)$ means that exactly $m$ of the $\{k_i\}$ indices are non-zero. First we use the symmetry of the $\{k_i\}$ indices and recast Eq. (B1) in the form

\[S^m_{N,m} = \binom{N}{m} \sum_{k_1 + k_2 + \ldots + k_m = n} \frac{n!}{k_1! \ldots k_m!}, \quad (B2)\]

where in the remaining $m$-fold sum all the indices are non-
zero. For the calculation of $S_{m,a}^N$ we will need the well-known summation formula for the multinomial coefficients

$$Z_{m,n} = \sum_{k_1 + \ldots + k_m = n} \frac{n!}{k_1! \ldots k_m!} = m^n. \quad \text{(B3)}$$

We show now that the following sum equals to $S_{m,a}^m$:

$$\left(\begin{array}{c} m \\
0 \end{array}\right) Z_{m,n} - \left(\begin{array}{c} m \\
1 \end{array}\right) Z_{m-1,n} + \left(\begin{array}{c} m \\
2 \end{array}\right) Z_{m-2,n} - \ldots + (-1)^m \left(\begin{array}{c} m \\
m \end{array}\right) Z_{0,n} = S_{m,a}^m. \quad \text{(B4)}$$

The first term contains all the coefficients, where any of the $\{k_i\}$ indices are zero. With the second term we subtract all the coefficients, where at least one of the $\{k_i\}$ indices is zero for sure. Coefficients, in which more than one, say $l$, indices are zero are subtracted $l$ times by the second term. The third term adds them again ($\frac{m}{l}$) times, since it represents all the coefficients with two or more zero indices. Following this induction, we see that coefficients with exactly $l$ zero indices are contained altogether with a factor

$$\left(\begin{array}{c} m \\
l \end{array}\right) - \left(\begin{array}{c} m \\
l-1 \end{array}\right) + \left(\begin{array}{c} m \\
l-2 \end{array}\right) - \ldots + (-1)^l \left(\begin{array}{c} m \\
l \end{array}\right) = 0. \quad \text{(B5)}$$

Hence, substituting Eqs. (B4) and (B3) into Eq. (B2) we arrive at the result

$$S_{m,a}^N = \left(\begin{array}{c} N \\
m \end{array}\right) \sum_{i=0}^m (-1)^i \left(\begin{array}{c} m \\
i \end{array}\right) (m-i)^n. \quad \text{(B6)}$$